MATHEMATICS

An Overview of Its History and Philosophy
From a Christian Perspective

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INTRODUCTION

It was the original purpose of this paper to seek out and investigate a unique Christian perspective of the body of concepts and practices which comprise mathematics. Such an investigation, we hoped, would lead to a unique Christian philosophy of mathematics which would quite naturally lead to distinctive practices in mathematical instruction and, to a degree, in application. It soon became evident that any claim to a unique Christian philosophy that would adequately explain the realities of mathematical experience would be difficult to substantiate.

Nevertheless, we cling to the ideal goal of the paper. But, rather than to adjust and reshape mathematics to satisfy the rigidity of a unique philosophy, we will attempt to develop a Christian philosophy that is capable of being enlarged and reshaped to accommodate new mathematical concepts and perceptions as they are encountered.

We admit at the outset that our efforts will be found wanting. At best it is our hope that this discussion will be enlightening, and that it will raise questions and criticism which will help us to gain a distinctive insight into the substance of mathematics, its history, its philosophy, and how mathematical knowledge is discovered, shared, and applied.

It would seem a good and necessary thing at this point to set forth a definition of mathematics that would leave no doubt concerning what mathematics is and what it is not. But when we consider the broad spectrum of what is called mathematics, and when we consider the often times unpredictable direction of the growth of mathematics, we must conclude that any attempt to define mathematics is to confine it. A carefully worded definition of mathematics of the early Babylonians or Egyptians, for example, would be totally inadequate for the mathematics of today. Many attempts have been made to write such a definition; however, they have been either too vague or too restrictive to be universally accepted.

We will consider three such definitions. The first two that we consider are extreme positions and should be considered "shots that straddle the target."

Bertrand Russell (1872-1970) defined pure mathematics as "the class of all propositions of the form p implies q, p and q being carefully specified." Russell began a treatise to provide a justification of his definition. Alfred North Whitehead (1861-1947) collaborated with Russell in this effort resulting in the production of their ***Principia Mathematica*** **—** a brilliant work of considerable influence. Most mathematicians would not accept this definition, for logic has a major role in discipline other than mathematics.

A second definition, which has more appeal to the applied mathematician, states that "mathematics is the language of experimental physics." Aside from the fact that this definition would require a definition of "experimental physics," it is incomplete, since much of mathematics is not applied to physics, experimental or otherwise.

A third definition, supplied by Davis and Hersh in their book ***The Mathematical Experience***, is more historically and intuitively appealing. "Mathematics is the science of quantity and space." This, they point out, "is a naive definition; however, if it is added that mathematics also deals with symbolism relating to quantity and space, the definition has a historical basis, and would serve as a start which would have to be altered from time to time to reflect the growth of mathematics."

Part 1
HISTORICAL SKETCH

Were we able to write a definition of mathematics, it would necessarily reflect our philosophy of mathematics. This philosophy should include distinct epistemological considerations, and it should be reflected in the way that we practice mathematics.

To many, mathematics appears rather cold, austere, and correct beyond a shadow of a doubt. Some, who have attempted to give a presentation of mathematics from a distinctively Christian point of view have pointed to the orderliness and correctness (to the point of infallibility) of mathematics as somehow reflecting the orderliness of the physical universe as it was created by the hand of God. Not true. The universe operates precisely according to certain laws established, no doubt, in the beginning when God created it. Mathematics, on the other hand, is not an infallible body of truths. In fact, mathematics, in the early stages of its history, was developed on a more or less empirical basis. It was made up of a series of disconnected simple rules based upon observations of success of applications.

Of the early civilizations that practiced mathematics, the Egyptians and Babylonians were the most prominent. Their mathematics was crude, disconnected and often times approximate. I Kings 7:23, for example, contains a passage which suggests that the Babylonians used the value 3 for the circumference/diameter ratio of a circle. When God spoke to His people concerning quantitative matters, He addressed them at their level of mathematical sophistication. It took centuries before the number system was expanded to include irrational numbers of which the actual value of the ratio of circumference to diameter is a member.

The Babylonians and Egyptians used, for the most part, inductive reasoning to develop their rules for mathematical applications. It should be noted here that, though the mathematics of these civilizations was not well developed, methods of numeration were invented. Sometime before 2000 B.C. the Babylonians developed a base sixty or the sexagesimal system of numeration. The Egyptians invented a base ten numeration system.

The mathematics of the Babylonians and Egyptians served as a prelude to the work of the Greeks, though it served as a shaky foundation on which to build.

The Greeks determined that mathematics must be a body of truths. How to find these truths and guarantee that they were truths evolved during the period from 600 B.C. to 300 B.C. The Greeks concluded that mathematics must deal first of all with abstract concepts, each of which was to embrace all physical occurrences of that concept. The number 2 is an abstraction which would embrace two of anything — two people, two feet, etc. While to the Egyptian a straight line was the result of drawing along a straight edge, to the Greek it was an idea. Mathematics, according to the Greek, would start with basic abstract concepts, the meanings of which would be “clear” to “all,” and which would be accepted as undefined terms. Beginning with these primitive terms, other concepts would be defined, and axioms (self evident truths) would be set down from which theorems were derived using deductive reasoning. This deductive method, when applied to any premise, yielded conclusions as valid as the premise. Using this method, referred to today as the axiomatic method, Euclid produced ***The Elements*** around 300 B.C. ***The Elements*** were comprised of a deductive chain of 465 propositions which embraced plane and solid geometry, number theory, and Greek geometrical algebra. This work was accorded the highest respect, and it superceded all previous efforts of the same nature.

The deductive method was a radical departure from previous mathematical methodology. To gain an appreciation of this, consider the following example: Suppose we were to determine the sum of the measures of the interior angles of a triangle. We might measure, as accurately as possible, the interior angles of a number of different triangles. After performing the experiment we would certainly conclude that the sum of the measures of the angles of a triangle is 180. This is correct, as we all know; however, our conclusion is unacceptable because it is based on inductive rather than deductive reasoning. This inductive method would have been the approach used by the Egyptians. The Greeks would have approached the problem somewhat as follows: Given the following axioms and assumptions in terms of the figure 2.1:

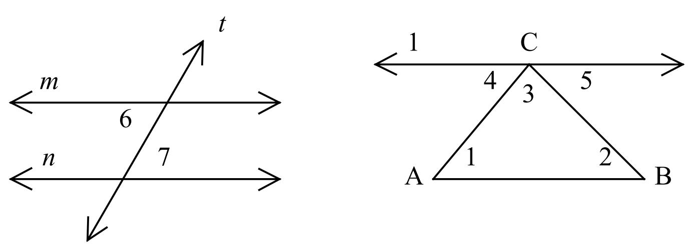
1) If line *t* intersects parallel lines m and n, alternate interior angles are equal.
2) The measure of a "straight angle" is 180.
3) The sum of the parts equals the whole.
4) Any expression can be substituted for its equal.
5) Line [1] intersects triangle ABC at point C and is parallel to segment AB.

1) m4 + m3 + m5 = 180 [axioms 2 and 3]
2) m1 = m4 and m2 = m5 [axiom 1]
3) m1 + m3 + m2 = 180 [axiom 4]

Thus the sum of the measures of the angles of a triangle is 180.

Our conclusion is true only if the axioms are true. This is all that is claimed for the deductive method.

Fig. 2.1. Angles 6 and 7 are alternate interior angles.



It should be noted here that, while the Greeks applied the deductive method in arithmetic as well as in geometry, the emphasis in using the deductive method was in the field of geometry. This tradition in mathematics persisted until very recent times.

It is not the purpose of this paper to give a detailed account of the history of mathematics, but rather a background sketch to aid in the consideration of certain of the various philosophies of mathematics. However, we ought not discuss the Greek influence on mathematics without mentioning the chief contributors. Three of the most outstanding Greek mathematicians were Euclid (c. 300 B.C.), Archimedes (287-212 B.C.), and Apollonius (c. 225 B.C.). Almost every significant subsequent development in geometry had its roots in the work of these three mathematicians. Euclid is probably the best known of the three through his work in ***The Elements***, which is considered the most influential single work in geometry in the history of the subject. Archimedes (287-212 B.C.) is considered by many to be one of the greatest mathematicians of all time. He anticipated concepts such as continuity, which has considerable importance in modern mathematical analysis. Apollonius (c. 300 B.C.) was a prolific writer, as were Euclid and Archimedes. He made many contributions to geometry that earned him the title "The Great Geometer." The terms "ellipse," "parabola," and "hyperbola" were originated by Apollonius. After Apollonius, the Greek influence in mathematics declined.

Euclid, Archimedes, and Apollonius lived during what is referred to as the Alexandrian Greek period. During this time, the Romans became a world power. Around the end of the third century, they conquered the Greek mainland and a number of Greek cities scattered through the Mediterranean area. Among those cities was the city of Syracuse in Sicily, where Archimedes spent most of his life. According to an account given by the historian Plutarch, a Roman soldier shouted to Archimedes to surrender, but the latter was so absorbed in studying a mathematical problem that he did not hear the order, whereupon the soldier killed him.

The final destruction of the great city of Alexandria is "credited" to the Arabs in A.D. 640. In a relatively short period of time the Arabs built a civilization and culture that lasted from about 800 to 1200. Realizing the value of the works that the Greeks had created, the Arabs translated the Greek works they could still find into Arabic. Among these were the writings of Aristotle, Euclid, Apollonius, and Archimedes. We are indebted to the Arabs, not only for preserving the mathematical contributions of the Greeks, but also for transferring some mathematical works from India. This included the invention of number symbols 1-9, 0, and the use of positional notations with base ten. This notation is still used today. The Hindus also created negative numbers.

Since the European civilization at this time was primitive, it is not surprising to find that mathematics was practically unknown in Europe until about A.D. 1100 — the end of the period in Europe known as the Dark Ages. During the Dark Ages the Arabs and Hindus were the chief custodians of mathematics. It is interesting to note that the term algebra is taken from the title of a book written by Alkhowarizmi, a ninth-century Arabian mathematician; and algorithm, now meaning a systematic process of calculations, is a corruption of his name. It should also be noted here that the Arabs should be credited for introducing trigonometry, both plane and spherical.

Toward the end of the eleventh century, Greek classics in mathematics began to filter into Europe. Christian scholars translated the ancient works into Latin. We may think of the twelfth century, in the history of mathematics, as the century of translations.

During the thirteenth century, universities were established at Paris, Oxford, Cambridge, Padua, and Naples. These universities became centers for the further development of mathematics. In about A.D. 1260, Johannes Campanus made a Latin translation of Euclid's ***Elements***; then in 1482 the first printed version of ***The Elements*** was made from this translation.

The fourteenth century was the century of the Black Death, which resulted in the death of more than a third of the population of Europe. During this century, the Hundred Years War, with the resulting political and economic upheavals, adversely affected the development of mathematics.

In the fifteenth century, with the collapse of the Byzantine Empire, resulting in the fall of Constantinople to the Turks in 1453, refugees fled to Italy, bringing with them many of the Greek classics. These Greek works, known to that time only through Arabic translations which were often inadequate, could now be studied from original sources. During this time mathematical activity was centered in the European cities of Nuremberg, Vienna, and Prague. The development of mathematics was not particularly rigorous by today's standards. It was influenced by the practical considerations of trade, navigation, astronomy, and surveying.

The development of arithmetic and algebra continued during the sixteenth century. During this time the language of algebra changed from rhetorical to symbolic form. This development influenced algebraic achievements in a positive way. Also it had a marked effect on the expansion of geometry. As is often the case, mathematical symbols have a power to facilitate the development of mathematical concepts beyond the expectations of their originators. An important achievement during this century was the discovery of the algebraic solution of cubic and quartic equations.

The seventeenth century was a period of extensive mathematical investigation. Many new fields of mathematics were discovered. Probably the greatest mathematical achievement of the century was the invention of the calculus by Isaac Newton and Gottfried Wilhelm von Leibniz. It is rather interesting to note that these two men invented the calculus independently. Also, to say that Newton and Leibniz invented the calculus is not entirely accurate, for much work had been performed prior to this time. There are two major concepts in the calculus: the derivative and the integral concept.

Leibniz introduced symbolism for the derivative still in use today. He also did extensive work with the integral. In fact, he discovered the theorem which is referred to today as "the fundamental theorem of calculus." This amazing theorem shows that certain "infinite sums" can be found by reversing the process of finding the derivative. Neither Newton's nor Leibniz's ideas were clear, and their proofs relied upon geometrical representations. There are many pitfalls in this approach as most modern mathematics students realize. It should be kept in mind, however, that the calculus involved new and subtle ideas which even the best of creative minds would not be expected to grasp immediately. Newton was certainly one of the greatest of mathematicians, and Leibniz ranks with Aristotle intellectually. We should view the development of the calculus, at this stage, as a creative effort which always precedes formalization and therefore lacked logical foundation.

The invention of calculus was not the only significant event of the seventeenth century. In response to Renaissance artists' and architects' efforts to produce more realistic pictures, projective geometry was invented. However, this was overshadowed by Rene Descartes' work in analytic geometry — a very significant concept. This should not be viewed as another branch of geometry, but rather a method of geometry which allows us to apply algebra to the study of geometry. At this point in the history of mathematics, algebra was not considered an independent branch of mathematics. It was regarded as a tool for analyzing geometric problems. After the creation of this analytic method for studying geometry, algebra was gradually converted into an independent branch of mathematics.

In the minds of the seventeenth century mathematicians, geometry as inherited from the Greeks had a secure foundation. The number system, algebra, and — as we have already seen — calculus had no such logical foundation. That this was the case caused some uneasiness however, because, for the most part, mathematicians of the seventeenth century had little concern for this foundational problem.

The question could be raised here concerning why, in the history of mathematics, there was such concern for the deductive development of geometry while, on the other hand, there was an apparent lack of concern for a logical development of algebra and the number system. In answer to this, it is suggested by Morris Kline, in his book, ***Mathematics — The Loss of Certainty***, that geometrical concepts are intuitively more accessible than those of algebra and the number. In geometry, diagrams aided in revealing structure; but these diagrams, when they became available, did not suggest a logical organization of the number system or of algebra which is built upon the number system.

We should not leave the seventeenth century without at least mentioning Pascal, best known for his work in probability theory, and Fermat, for his work in number theory. Fermat is probably best known for his "last theorem" which asserts that there exists no three whole numbers (or fractions) x, y, and z such that $x^n + y^n = z^n$ if "n" is a whole number greater than 2. Fermat stated that he possessed a "marvelous proof" for his assertion; however, the proof has never been found, and in the last 300 years no one has been able to prove or disprove his claim. Recently (1989), a Japanese mathematician claimed to have proved the theorem, using methods that would not have been available to seventeenth century mathematicians. At first glance, it appeared that his proof was sound; however, upon closer scrutiny flaws were found in his proof. So Fermat's "last theorem" still remains to be proved.

Eighteenth century mathematicians attempted to secure the foundations of the calculus on a logical basis. This effort was for the most part unsuccessful since the calculus has its basis in arithmetic and algebra which lacked secure foundations. Despite this fact, eighteenth century mathematicians extended calculus, and derived new concepts such as infinite series, differential equations, differential geometry, calculus of variations, and theory of functions. The calculus, along with these extensions, are collectively referred to as analysis today.

The fact that the calculus gave birth to new concepts in mathematics, and that these investigations lacked rigor, became a growing concern of many mathematicians and philosophers of the day. One of the strongest criticisms was registered by the philosopher Bishop George Berkeley (1685-1753), who feared that the mathematically inspired philosophy of mechanism and determinism would eventually have an adverse effect on theological reasoning. Though his concern seemed not to be so much for sound mathematics as it was for sound theology, his criticism produced results in the mathematical community. Of those who responded to Berkeley's criticisms, Euler produced one of the more useful works in that he freed the calculus from geometry and based it on arithmetic and algebra. This was the first step to the justification of the calculus on the basis of number. Euler's approach was formalistic; however, his efforts were lacking.

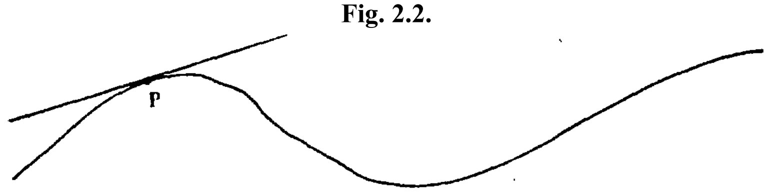
Lagrange (1736-1813), a mathematician of considerable stature, also sought to rigorize calculus. His approach was to reduce the calculus to algebra. Though he contributed significantly in his work in differential equations and the calculus of variations, his efforts to secure a foundation for the calculus were unsuccessful. At a time when a definition for the limit concept was needed, Lagrange dispensed with the limit in his effort to reduce the calculus to algebra. Although Lagrange contributed significantly to analysis, there were some glaring inadequacies in his efforts to secure a logical foundation for analysis. Though there were some who accepted Lagrange's foundational work, it became clear that proper foundations for analysis had not been secured. Lagrange recognized this and proposed in 1784 that a prize be awarded in 1786 for the best solution to the problem of the infinite. A number of mathematicians attempted to solve the problem, but none were successful. At this point, the general view of the calculus could be summarized by Voltaire's description of the calculus as "the art of numbering and measuring exactly a thing whose existence cannot be conceived." At the end of the eighteenth century, efforts to build a logical foundation for analysis were in a confused state.

At the beginning of the nineteenth century, mathematicians faced a rather interesting paradox. Although mathematics had a very shaky, and in some areas an almost nonexistent logical foundation, it was a powerful and accurate tool in predicting and representing physical phenomena. Evidently intuitive insight of the powerful intellects of the day sufficed when rigorous logical foundations seemed unattainable. Because of the effectiveness of mathematics in application and for other less obvious reasons, many mathematicians had little regard for proofs. These include mathematicians of such stature that they are known to every undergraduate student of mathematics today. Arthur Cayley (1821-1895), inventor of matrix algebra, stated a theorem known to mathematics students today as the Cayley-Hamiiton theorem. Concerning the theorem he said, "I have not thought it necessary to undertake the labor of a formal proof of the theorem in the general case…" The well-known British algebraist James Sylvester (1814-1897) was known to have commented in his lectures, "I haven't proved this, but I am sure as I can be of anything that it must be so." Often, as Sylvester himself confessed, it wasn't so.

Numerous errors were introduced into mathematics because of this rather cavalier attitude toward proofs. Besides this, there were other blunders because mathematicians often used intuitive evidence in their proofs. Consider the following example:

> The concepts of a continuous function and derivative of a function are fundamental in analysis. Intuitively a continuous function is one that can be represented by a curve drawn without raising a pencil. The geometric meaning of the derivative of such a function is the slope of the line tangent at any point P of the curve (see fig. 2.2).

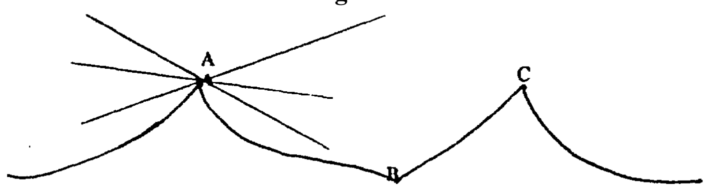
Fig. 2.2.



Intuitively it would seem clear that each point on the curve would be associated with a distinct tangent line with a unique slope. Therefore, on an intuitive basis, we could conclude that a continuous function has a derivative at every point. Unfortunately this false assumption is made by many undergraduate students of mathematics today. Worse yet is the case of a graduate student who recently, during his oral examination for his master's degree, asserted that continuous functions are differentiable. (He flunked his orals.)

As Figure 2.3 "shows," continuous functions that can be represented by curves with "corners," as at A, B, and C, do not have derivatives at such points.

Fig. 2.3.



In the late nineteenth century, Weierstrass devised a continuous function with no derivatives at any point. The standard of rigor today would not allow such intuitive evidence based upon a geometric figure.

How could the leading mathematicians of the day make such gross errors? With reference to the present example it should be realized that functions that give rise to a curve such as that in Figure 2.2 were much more common to the experience of the eighteenth and nineteenth century mathematicians than were functions associated with a curve such as the one in Figure 2.3. There can be no doubt that application motivated mathematical practices.

A major reason for frustrations in efforts to logicize new mathematical concepts can be seen in a subtle change in the nature of mathematics — a change not fully grasped by even the best minds of the day. Mathematical concepts, through about the fifteenth century, were abstractions from experience. But with the introduction of concepts such as the derivative and the integral, though they can be viewed intuitively through their applications, are not abstractions from experience, but rather concepts devised by the mind which have numerous applications. More and more it became the case that mathematicians were inventing concepts rather than abstracting ideas from experience. Because they failed to see this change in character in mathematics, they failed to realize that "proofs" based upon experience were inadequate.

By the second decade of the nineteenth century a concerted effort had begun to overhaul the logical foundations of mathematics. There was a wealth of mathematical talent available to undertake this ambitious plan. Augustine Cauchy (1789-1857) should be noted for his role in rigorizing the calculus. He determined that the calculus should be founded on the limit concept. Karl Friedrich Gauss (1777-1855), considered by many to be the greatest mathematician of all time, made considerable contribution in just about every area of mathematics. Other nineteenth century mathematicians who should be given special mention for their contributions to moderm mathematics are Riemann, Dedikind, Weierstrass, and Hilbert, each of whom contributed significantly to mathematical analysis as well as to foundations of mathematics.

The work of the nineteenth century mathematician Georg Cantor is of particular interest. Cantor profoundly influenced modern mathematics through his work on theory of sets — a concept which forms one of the characterizing features of modern mathematics, though this was not envisioned by many mathematicians in Cantor's time.

He was referred to as a charlatan. One mathematician predicted, "Later generations will regard set theory as a disease from which one has recovered." But he also had some notable supporters. Russell saw Cantor as "one of the great intellects of the nineteenth century." Hilbert stated, "No one shall drive us from the paradise which Cantor created for us." He saw Cantor's work as "the most admirable flower of the mathematical intellect and one of the highest achievements of purely rational human activity." We shall see later that these various views on Cantor's work represent differing views of the philosophy of mathematics.

We have already noted that set theory plays an important role in modern mathematics. But it was the discovery of contradictions in the theory of sets that provided the chief impetus for investigations concerning the foundations of mathematics.

The division of the time scales of history into eras is arbitrary to a degree. So we will consider the modern mathematics era as that period of time that began approximately one hundred years ago with the efforts of Cantor and others to build the real number system on a logical foundation. It was to obtain a more penetrating description of the structure of real numbers that the theory of sets was born. The concept of sets established its central role in modern mathematics, however, because of its use in investigating the nature of numbers. Today sets are often used in the classroom to distinguish between the various kinds of numbers and to investigate their relationships. For example, if Q is the set of rational numbers and Ir is the set of irrationals, then Q $\cup$ Ir = R — the set of real numbers.

A set could be defined simply as a collection of objects. In order for a set to have value in application, we must extend the definition to include that these objects must have something in common. That is, it must be clear whether or not a given object is in a given set. In fact, sets are used to distinguish certain objects from all other objects.

Sets, usually designated by capital letters, are either finite or infinite. For example, consider a set A which consists of the even numbers between 1 and 11. The set A is finite since the elements in the set could be counted and a natural number assigned to it (5 in this case). The set B, which consists of the even numbers, is an infinite set. The idea of one-to-one correspondence is of importance in set theory. Two finite sets can be placed in a one-to-one correspondence if they have the same number of elements. Using the common notation for sets, let A = (1, 2, 3, 4); B = (2, 4, 6, 8). A one-to-one correspondence could be established as follows:

1	2	3	4
2	4	6	8

We could use a set which demonstrates this correspondence: ([1, 2], [2, 4], [3, 6], [4, 8]).

Cantor extended this notion to infinite sets. If A is the set of natural numbers, and B is the set of even positive numbers we could obtain the following one-to-one correspondence:

1	2	3	4	5	...
2	4	6	8	10	...

Since sets A and B have a one-to-one correspondence, Cantor reasoned that they have the same "number" of objects. Cantor referred to these "numbers" as transfinite numbers. The set of natural numbers, and sets that can be put in a one-to-one correspondence with it, have the same number of objects. This number he denoted by $\chi_{.}$ (aleph-null), the "smallest" of transfinite numbers. Though it is a bit tricky, it can be shown that the set of natural numbers can also be put into one-to-one correspondence with the set of rational numbers; however, no such correspondence exists between the natural numbers and real numbers. The set of all real numbers is "larger" than the set of natural numbers. He denoted the transfinite number of elements in this set with the letter "c."

Suppose A is a finite set consisting of the numbers 1, 2, 3 — i.e., A = (1, 2, 3). We could think of the subsets of A — that is, sets which contain one or more elements of A. The following list includes all the subsets of A: [1], [2], [3], [1, 2], [2, 3], [1, 3], [1, 2, 3], ϕ. The latter two subsets, namely the set itself and the null or empty set, are referred to as improper subsets of A; the others are proper subsets. The empty set ϕ is the set which contains no elements. For logical precision it is considered a subset of every set. This may seem a strange requirement, but we will not undertake to discuss the necessity of the requirement other than to point out that a set is the union of all its subsets. The inclusion of ϕ as a subset does not violate the above statement. For those who are unfamiliar with the algebra of sets, we define the union of two or more sets to be the set containing elements in at least one of those sets. We use the symbol "$\cup$" to denote "union" — for example, (2, 3) $\cup$ (1, 3) = (1, 2, 3). An alternate definition could be stated thus: If the set S = A $\cup$ B then if *x* belongs to A or if *x* belongs to B, then *x* belongs to S. The connective "or" is used in mathematics (and in logic) in the inclusive sense.

Now if we count the subsets of A = (1, 2, 3) we find that there are 8 subsets. But $8 = 2^3$. This suggests that there are 2^n subsets associated with a finite set containing *n* elements. That this is true can be easily proved. Another observation to be made here is that the number of subsets of a given set is always greater than the number of elements in the set. Cantor proved that this is also true in the case of infinite sets. Recall that the "numbers" of elements in the infinite set of natural numbers is $\chi_{\bullet}$. Cantor denoted the number of subsets of the set of natural numbers with the symbol $2^{\chi_{\bullet}}$. It is not difficult to see that $2^{\chi_{\bullet}} = c$ — the transfinite number of elements in the set of real numbers. We must keep in mind here that transfinite numbers do not represent quantity as we usually think of quantity. Cantor asserted that there exists no transfinite numbers between $\chi_{\bullet}$ and $2^{\chi_{\bullet}}$. This is referred to as Cantor's continuum hypothesis. A geometrically orientated statement of Cantor's hypothesis is stated as follows: Is there on a line segment an infinite set of points that is not equivalent to the whole segment, and also not equivalent to the set of natural numbers? Cantor said no. In 1900 David Hilbert drew up a list of unsolved problems. Cantor's continuum problem was first on the list. This list was limited to those problems that, in Hilbert's view, were the most important and pressing problems of the day; otherwise the list would have been endless.

It should be noted here that Cantor's transfinite numbers, which regard infinite sets as totalities that can be comprehended by the mind, was opposed to long-standing conceptions concerning the infinite. From the time of Aristotle, mathematicians had made a distinction between the actual infinite and the potential infinite. The actual infinite with respect to the set of natural numbers, for example, embraces its entirety. That is, there is no natural number not included in the count denoted by the transfinite number $\chi_{\bullet}$. On the other hand, the set of natural numbers is regarded as potentially infinite if we consider that no matter how large a number "n" that we choose, there are natural numbers greater than "n" such as n + 1, n + 2, etc. The potentially infinite cannot be completely comprehended. Cantor realized this and said in 1883, "I place myself in a certain opposition to widespread views on the mathematical infinite and to oft-defended opinions on the essence of number."

Initially Cantor's work with infinite sets was not considered of particular consequence; however, by the beginning of the twentieth century, set theory was applied in many areas of mathematics. It had already by that time become an almost indispensable tool in mathematics. We have already noted that it was used to secure a foundation for the real numbers. His theory of infinite sets was also used in the generalization of integral calculus. Another interesting example of the application of Cantor's transfinite numbers had to do with the proof of the existence of transcendental numbers. Until the nineteenth century it was not known whether any numbers are transcendental. A number is considered algebraic if it satisfies a polynomial equation of the form:

$$a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 = 0$$

where the a_k's are integers. A number that is not algebraic is called "transcendental." An interesting feature of Cantor's proof of the existence of transcendental numbers is that existence was proved without exhibiting a single one. Furthermore, Cantor showed that "most" real numbers are transcendental. Some examples of transcendental numbers are π, 2^{π}, log 3, etc.

By the end of the nineteenth century there was a general feeling of confidence in the new foundations in mathematics and in mathematical rigor. Though Cantor's theory of infinite sets was by no means universally accepted, it had, by this time, been widely applied in mathematical investigation. Therefore, it was more than a little disconcerting when Cantor himself discovered a flaw in his theory of transfinite numbers which later became known as "Cantor's paradox." This paradox came to light when Cantor thought to consider the set of all sets. The transfinite number associated with such a set must be the largest that can exist. However, Cantor had already proved that the set of all subsets has a larger transfinite number than the set itself. Hence there must be a larger transfinite number than the largest one. Stated in another way, Cantor's theory had assumed that every set has a cardinal number. Now consider the question: Does the universal set of all finite and infinite cardinal numbers have a cardinal number or not? Cantor's theory assumes so, hence the answer would be yes. Yet the answer cannot be affirmative, for the number of all cardinals must be larger than any cardinal number. We have a contradiction.

Mathematicians of the early twentieth century recognized that this paradox in various forms was present in their work and also in older, supposedly well-established mathematics. This was but the beginning of the well-known "crisis" in mathematics that had considerable impact on the mathematics of the twentieth century. Thereafter much of the work in the foundations of mathematics resulted from this paradox, along with other contradictions in the theory of sets. Besides this, certain rules of logic were being challenged. One such rule is the "Law of the Excluded Middle," which simply asserts that a statement

is either true or false, but not both. There was also concern, particularly on the part of intuitionists, with respect to impredicative definitions. An impredicative definition is one which describes an object in terms of a class to which it belongs. While some impredicative definitions are innocuous, others are not. For example, the definition of the least upper bound is impredicative. The least upper bound of a set is the smallest of all upper bounds of the given set. This definition is used in most mathematics textbooks. Now consider the statement of the law of the excluded middle, which can be stated thus: “Every proposition is either true or false.” But the law itself is a proposition and therefore could be false, although its intent is to affirm a true law of logic. These are but a few of the difficulties that troubled those mathematicians and philosophers who were concerned with foundational problems in mathematics at the turn of this century. The resulting efforts to solve these inconsistencies imposed new standards of rigor, which marks twentieth century mathematics.

Remedies for this “crisis in foundations” were proposed by various schools of thought. These remedies, of which “logicism,” “constructivism,” and “formalism” had the most impact, will be discussed in the next section. The thing to keep in mind here is that mathematics involves much more than foundational problems. Most mathematicians, while they may have a concern for foundational problems, basically have confidence in mathematics and continue to do mathematics, extending it and creating new concepts as though its foundations were perfectly secure. Were this not the case, mathematics would have stagnated. Nevertheless, foundational concerns influenced twentieth century mathematics at all levels of practice, resulting in an axiomatic method which allows for less appeal to intuition (except as an aid to discovery).

To provide an overview of the development of mathematics in the twentieth century would be very complex and beyond the scope of this paper. We will briefly point out some features that distinguish modern mathematics, and call attention to people and events that influenced mathematical practices.

A review of the history of mathematics reveals that periods of research and extension were followed by periods of review and synthesis during which concepts in mathematics were consolidated. Mathematics today is in a state of rapid development and expansion. At the same time, there is greater concern for precision and economy of thought. The methods have become more abstract, so that the same concepts can be applied to problems of widely varying-kinds. This is due to the fact that the axiomatic method has penetrated every area of mathematics. This has been the unifying factor in mathematics. The tendency is toward generating theorems of sufficient generality to be applicable to various disciplines.

Technology, particularly computer technology, has had a profound impact on mathematical research and applications. The computer-aided proof of the four-color conjecture is but one example of the application of computers in pure mathematics. Numerical approximations are a direct result of the computer revolution.

Growth in mathematics doesn’t just happen, but rather is the result of the thoughts and actions of people.

The trends in mathematics in the United States were influenced by a number of people. E.H. Moore was a mathematician with an intense interest in mathematics education. Moore, who taught at the University of Chicago, was active in the formation of the Mathematical Association of America and the National Council of Teachers of Mathematics.

An event that influenced mathematics in the second quarter of the twentieth century was the founding of the Institute for Advanced Study in Princeton, New Jersey. Among the charter members were Albert Einstein, a brilliant mathematician best known for his “theory of relativity”; John vonNeumann, probably the last of the mathematical universalists who made significant contributions in many areas of mathematics including refinements to Cantor’s theory of transfinite numbers; and Kurt Godel, well known for his work on Cantor’s continuum problem. This institute had a profound effect on the production of mathematics in the United States and throughout the entire world.

The Second World War had a considerable impact on the mathematical world. Mathematicians were called upon to solve a variety of problems of both a physical and social nature. These applications resulted in accelerated development in established disciplines, and motivated investigation into new fields such as “game theory” and-“information theory.” Significant developments in computers and computer theory has almost all been post-Second World War.

The last quarter of the twentieth century has witnessed the new phenomenon of government offering lucrative contracts and grants for research in the pure sciences, usurping what had been the prerogative of academic institutions. As a result, there was an acceleration of mathematical invention. Industry began to draw many mathematicians away from the academic life, with the result, that the demand for mathematicians far exceeded the supply.

Although mathematics curricula at the graduate level had kept pace with the newer and more powerful methods in mathematics, the mathematics curricula at more elementary levels remained static. This was particularly true of the elementary and secondary levels.

The first efforts to overhaul the mathematics curriculum were directed toward the high school college-preparatory programs. In 1952, Professor Max Beberman headed the “University of Illinois Committee on School Mathematics,” which produced experimental material which incorporated new approaches, emphasizing comprehension of the underlying principles of mathematics. After 1960 these experimental materials were published as texts. After the completion of this initial work, the committee worked on materials directed at the elementary school level.

In 1955, the C.E.E.B. (College Entrance Examination Board) began a program to improve high school mathematics instruction. This board established its own commission on mathematics. In 1959 they issued their report, “Program for College Preparatory Mathematics.” Included in this report were samples of their recommended methodology. These appendices to the commissions report were written as a guide for teachers, and not intended to be used as a textbook.

In the fall of 1959, Russia launched their first “Sputnik.” This event was significant in that it convinced the government that Russia must be ahead of us in mathematics and science. Government agencies and foundations began to make available funds which were to be used to improve mathematics and science curricula and to train teachers of mathematics and science.

With all this money available, other groups got into the act and began producing new mathematics materials.

In 1958, the American Mathematical Society, a research organization, became interested in setting up a new mathematics curriculum. It established a committee called “The School Mathematics Study Group” (S.M.S.G.), headed by Professor Edward Begle from Yale University. This group began by writing materials for junior high and high schools, then extended its efforts to provide curricula for the elementary grades. These materials had wide circulation and were used in many school systems. The S.M.S.G. materials were excessively wordy and wearied the minds of most students; however, many textbooks were written based upon recommendations of the S.M.S.G.

These and other study groups had considerable impact upon the content of mathematics textbooks and upon the way mathematics was presented in the classroom. The new curriculum still included traditional subject matter with an emphasis on the deductive method in its presentation. It also offered some new content. The most notable of these is set theory. As Kline puts it, “Set theory is now taught from kindergarten up, as though students would starve, mentally at least, if they did not have this diet.”

Initially most educators were enthusiastic about the new mathematics. However, it became increasingly evident that the “new math” had some flaws. Many of the same people who had enthusiastically endorsed the new approach began to clamor for a return to the basics. The pendulum began to swing away from the deductive approach that characterized much of modern mathematics instruction back toward the behaviorist approach. Mathematics instruction in elementary and high school was in a confused state. The National Advisory Committee on Mathematical Education (NACOME) called mathematics teaching a troubled profession.

Some of the criticisms of the new math were unfounded, but others were valid. Some of the criticism came from laymen, people in the community who had little or no understanding of the scope and goals of the new mathematics programs. Employers, notorious for their criticism of new, young employees, blamed the new math program when these employees seemed to lack skills in their (the employers’) own little corner of mathematical applications, conveniently forgetting that they themselves had at one time lacked familiarity with these skills. Perhaps the problem was not so much a lack of mathematical ability on the part of the young employee as it was a lack of patience on the part of the employer.

We sometimes make invalid judgment as a result of incorrect perception of statistics. It was noted that there was a significant drop in scores in certain tests. For example, there was a drop of about three points on the ACT. The period of this decline coincided with the introduction of the new math program in many schools. Many saw a cause and effect relationship here. Though there may have been some validity to this conclusion, it should be noted that there were similar declines in scores in other disciplines as well. Should the new approach in teaching mathematics be blamed for the decline in English scores? There are other possible reasons for the drop in scores. In the first place an ever-greater diversity in the student population was tested in the ACT. Given this fact, we would expect the mean to go down and the standard deviation to increase. This is exactly what happened. Also during this time there was a general social unrest that seemed to be accompanied by a decline in standards of discipline and respect for authority. The discipline in many schools eroded to a point that many teachers were unable to hold a meaningful class. Because of this, it is difficult to separate out the impact of the new math program.

There were some objections to the new methods that seemed to have some validity:

1. The new mathematics puts too much stress on the deductive approach. Even modern mathematicians develop new concepts using intuition and inductive reasoning before securing their theorems deductively.
2. Too much pretentious symbolism and terminology is employed. In making the language more precise, the meaning became less clear to the student. This serves only to confuse the student who becomes so bogged down in symbolism that he loses sight of the concept under consideration.
3. Students often lack the mathematical maturity to appreciate the rigor of the new approach. What may be good mathematics from a mathematician's point of view cannot be appreciated by the novice.
4. Some of the new content was inappropriate. It is doubtful, for example, that children in the early elementary grades appreciate the concept of functions and functional notation.

There was considerable debate over these and other objections. There are the extreme positions - those, on the one hand, who would favor a more formalistic, rigorous presentation, and, on the other hand, those who would throw out the new math entirely and go back to the old traditional cook-book type texts in which mathematics was presented as a collection of techniques to solve certain problems.

For the most part, educators have avoided these extremes, rejecting the excessively formal, but retaining the essential elements of modern mathematics.

A perusal of the latest literature reveals an altered emphasis in content and processes. These changes should be viewed as a cautious "midcourse correction" rather than an altogether new approach. The purpose is to improve mathematical instruction so as to strengthen conceptual and procedural knowledge, and to make use of computer related technology. The following list summarizes the new trends.

1) conceptualization of mathematical topics
2) improvement in problem-solving strategies
3) increased emphasis on graphical representation and interpretation
4) increased use of calculators and computers
5) more attention given to mental arithmetic and estimation techniques
6) more time given to plane and solid geometry
7) earlier introduction to algebraic concepts and symbolism

Though there have been disappointments, there has been a tremendous growth in mathematics in the twentieth century. This is also true of the quality of education in mathematics. The student of mathematics today stands head and shoulders above his counterpart of a few decades ago. There was a time when the undergraduate program in American universities and colleges did not go beyond calculus. Today it is not uncommon to find calculus courses taught in high schools.

Much of the growth in twentieth century mathematics has been from within, yet it has found application in other sciences. This has happened often. Pure mathematical concepts, though seemingly not motivated by application, at some later date have value in solving problems completely unknown to the inventors of those concepts. Some examples of this are Riemann's non-Euclidean geometry in the theory of relativity, and group theory applied in quantum theory in physics. Whitehead had observed this and commented: "Nothing is more impressive than the fact that mathematics withdrew increasingly into the upper regions of even greater extremes of abstract thought, it returned back to earth with a corresponding growth of importance for the analysis of concrete fact... The paradox is now fully established that the utmost abstractions are the true weapons with which to control our thought of concrete fact."

In the past, mathematics was created to solve problems of nature. The true mathematician, however, was not satisfied with solving the immediate problem, but, once having solved the problem, extended and generalized his work so as to have the broadest possible application. The great mathematicians of the past tended to be universalists. Because of the growth and complexity of mathematics today, mathematicians along with their colleagues in other disciplines tend to be specialists.

There is, in the minds of some, a real concern for what they consider a disproportionate tendency toward pure mathematics. Morris Kline contends that "mathematicians have abandoned science." George Birkhoff wrote, "It is to be hoped that in the future more and more theoretical physicists will command a deep knowledge of mathematical principles; and also that mathematicians will no longer limit themselves so exclusively to the aesthetic development of mathematical abstractions." John Synge was also concerned about the direction of mathematics toward the abstract. He wrote, in 1944, "Most mathematicians work with ideas which, by common consent, belong definitely to mathematics. They form a closed guild. The initiate forswears the things of this world, and generally keeps his oath. Only a few mathematicians roam abroad and seek mathematical sustenance in problems arising directly out of other fields of science... If mathematicians have really lost their old universal touch — if, in fact, they see the hand of God more truly in the refinement of precise logic than in the motion of stars — then any attempt to lure them back to their old haunts would not only be useless - it would be a denial of the right of the individual to intellectual freedom."

There was a concern that mathematicians would ascend their ivory towers, taking their mathematics with them, resulting in an ever widening gulf between pure mathematical research and applications in other sciences. Whether or not this represents a problem today is difficult to say. It would seem that there is a need for the purist, for the sake of continual growth and refinement of mathematics. We also need those mathematicians who are more inclined to application. And there is a need for what we will call the mathematical technician — the person who possesses the training and ability to apply the mathematical inventions of others, but who does not necessarily grasp the mathematical content. Perhaps we need in addition a concentrated effort to provide an instrument of effective communication between these groups.

There is direction in the growth of the multifaceted mathematics of today. But this direction is provided by the dictates of application and from within mathematics itself. It would seem that the direction and emphasis in the growth of this most fascinating field of intellectual endeavor should be given, at least in part, by those who practice mathematics at one level or another. If mathematics is the invention of the human mind — a question that will be discussed in the next section — then we should exercise some responsible control over that which we have invented. We should ask and seek answers to the question: What kind of mathematics should we be doing? The Christian should always be ready to give an account of himself, for the way he uses his time and his talents. This is not to stifle creative effort, but rather to give direction to those efforts. We are not free to practice mathematics, without concern for the results of our efforts, though we realize that there will always be the unpredictable in any mathematical research, as in the case in any human endeavor.

Part 2
PHILOSOPHY OF MATHEMATICS

The philosophy of mathematics as we view it in its historical context seems somewhat fragmented. The philosophy of twentieth century mathematics is identified with foundational problems in mathematics proper. It became evident near the beginning of the twentieth century that there were some glaring inadequacies in the logical foundations of mathematics. This "crisis in the foundations" came to light as a result of various inconsistencies discovered between Euclidean and non-Euclidean geometry, and as a result of contradictions in Cantor's set theory. There were of course other contributing factors, but these stand out as immediate contributors to the so-called crisis. From a historical perspective, this crisis, and the resulting identification of mathematical philosophy with foundational problems, was the consequence of a long-standing discrepancy between the ideal of mathematics and actual mathematical practices.

Any efforts to consider a Christian philosophy of mathematics would require some understanding of the background and development of major philosophies. We will attempt to provide an overview of the development of mathematical philosophy here.

The philosophy of mathematics dates back to the classical Greeks who expounded the philosophy of the mathematical design of nature. An understanding of mathematics through deductive reasoning would yield an understanding of nature. Concerning the existence of and acquisition of mathematical knowledge, two distinct views can be distinguished: there are the Platonists who believe that mathematics has independent existence apart from the human mind, and, on the other hand, there are those who believe that mathematics is entirely a product of the human mind.

The former "independent existence" philosophy was named after Plato (427-347 B.C.), who founded the academy in Athens which endured for 900 years. Platonists distinguish between the world of things and the world of ideas. The material world which can be perceived by our senses is imperfect and subject to change. About the material world we can have only opinions, while the ideal world is absolute and unchanging. The material world, the Platonists would say, is only imperfect realization of the ideal world. They believe that, though the concepts and properties which make up mathematics have independent existence, they can be apprehended by the human mind. Mathematics therefore, according to Platonists, is discovered, not created. What evolves is not mathematics but our knowledge of mathematics.

The question arises concerning where this objective, unique body of mathematics of the Platonists resides. Also there are epistemological considerations. Plato's answer to these concerns is based on his theory of anamnesis. He believed that man had experience as a soul in another world before coming to earth, and that the soul had but to be stimulated to recall prior experience in order to know that axioms of mathematics represented truth. He believed that a few "penetrating glances" at the physical world suggest basic truths with which reason could carry on unaided. Most Platonists (a name now used to refer to anyone who believes in the independent existence of mathematics apart from the human mind) do not share Plato's belief in anamnesis.

It is assumed by the Platonists that their mathematics is unflawed. What then about the existence of false propositions and contradictions in the Platonists' independent body of mathematics? Platonists would answer that such propositions and contradictions arise only because of man's inadequacy in his efforts to perceive the truth contained in this body of mathematics. But of course this absolute and perfect body of pre-existing mathematics is of very little use to its if we are misguided in our perceptions of it.

The Platonic philosophy of mathematics has persisted to the present time in various forms. Some God-fearing mathematicians saw mathematics as an intangible part of God's creation and therefore absolute and perfect. For this reason they had absolute confidence in their discoveries, evidently feeling that their insight as well as their mathematics was infallible.

The following quotes are included to give the reader some feeling for the various shades of the Platonic philosophy of mathematics:

"Although the truth is not yet known to us, it pre-exists, and inescapably imposes on us the path we must follow."

Jacques Hardamond (1865-1963)

"I believe that the numbers and functions of analysis are not the arbitrary product of our spirits; I believe that they exist outside of us with the same character of necessity as the objects of objective reality; and we find them or discover them and study them as do physicists, chemists, etc."

Charles Hermite (1822-1901)

The following quote is taken from remarks made by Kurt Godel, a modern mathematician discussed earlier:

"It seems to me that the assumption of such objects (sets) is quite as legitimate as the assumption of physical objects and there is as much reason to believe in their existences. They are in the same sense necessary to obtain a satisfactory theory of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions..."

Kurt Godel (1906-1978)

One notable Platonist, Georg Cantor (1845-1918), whose work we discussed earlier, believed that mathematicians are discoverers of concepts and theorems rather than inventors. Cantor regarded himself a reporter and secretary.

A second view, that mathematics is entirely a product of human thought, is held by mathematicians of several schools of thought, and can be traced back to Aristotle. There are those who believe that the source of mathematics is in the organizing power of the mind. According to them, mathematics originates in the physiology of the mind. More prominent is the idea that mathematics originates in the activity of the mind. "The mind," they say, "has the power to devise structures based upon experience and provides a mode of organizing them." Among those who assert that mathematics is a creation of the human mind, there is division in the camp. There are some who maintain that truth is guaranteed by the mind, while others maintain that, since the human mind is fallible, truth is not necessarily achieved. Again we have the epistemic concern. How valid is the mathematics we have created? And how do we determine validity? In response to such questions dealing with epistemological principles and mathematical practices, several schools of thought have emerged. It is interesting to observe that the concept of the infinite is the distinguishing factor among the various philosophies which emerged at the beginning of the twentieth century. These schools of thought will be discussed in detail when we consider twentieth century philosophy of mathematics.

The European absorbed the Greek works and philosophy. They pursued the study of nature by seeking underlying mathematical design. There was a conflict here, however, for the Greeks believed in the mathematical design of nature, while the Europeans believed that God created the universe according to His own plan (the laws of nature, therefore, originated from Him). Since the doctrines of the Catholic church did not allow for a mathematical design of nature, a new doctrine that God designed the universe mathematically was added, thus fusing the Greek philosophy and the doctrines of the church. (There is an interesting parallel here with present circumstances — with a difference. Instead of adding doctrines of science to those of Scripture, science is used to interpret Scripture so that those doctrines of the church not found in accord with scientific "findings" are simply supplanted by the philosophy of science. We think of the literal rendering of the Genesis account of creation as an example.) Viewing creation as a kind of mathematical venture provided incentive for mathematicians of the sixteenth, seventeenth, and eighteenth centuries to search for mathematical laws of nature to provide a clearer understanding of God's will and His creation.

Mathematics was viewed as the medium through which God's creation could be properly perceived. If this was true, then mathematics either belongs to an intangible part of His creation and is related to the tangible in such a way that we, by discovering mathematical concepts, accordingly discover something of God's mathematical design of nature, or else mathematics is somehow "embedded" in the human mind so that the mind can intuit mathematical truths. According to this latter view, God created the human mind so that it had the intuitive power to apprehend basic truths immediately. From these basic truths the mind could deduce further truths which would enhance our understanding of nature.

One would think that there would be harmony among Christian mathematicians with respect to these two views. In fact, this is not the case. It has always been true that mathematicians, even those who belonged to the same school of thought concerning foundations, with the exception of intuitionists, have been divided over whether mathematics pre-exists and is discovered (Platonism), or is generated by the mind. We have already listed some of the Platonists — a list which includes some of the greatest mathematical minds, including such notables as Gauss, Leibniz, and Newton.

Descartes deserves special attention here because of the influence of his philosophy on seventeenth century thought. He believed that concepts of mathematics are innate in our minds and that they are placed there by God. Along with other philosophers of the day he believed that God designed the world mathematically, and therefore the "laws of nature" are invariable. He also believed that God fixed these "laws of nature" at the time of creation — a contradiction of the prevailing belief that God continually intervenes, making adjustments here and there in the functioning of His creation. With respect to the study of nature, he asserted that mathematics was sufficient to explain all phenomena. Since all truths of nature can be deduced from mathematics, no further principles of physics were needed. He made a rather strong statement concerning this in his ***Principles of Philosophy*** (1644): "...I do not think that we should admit any additional physical principles, or that we have the right to look for any other."

Descartes was the founder of the philosophy of mechanism — a philosophy that declares that all natural phenomena, including the human body but excluding the soul, reduce to motions of particles which obey the laws of mechanics. Although he did perform experiments in various fields of science, and although he did draw some significant inferences from those experiments, he nevertheless believed in ***a priori*** truths, and that the intellect is capable of attaining infallible knowledge.

Another influential mathematician and philosopher of the seventeenth century, Blaise Pascal (1623-1662), gave considerable credence to intuition, as did Descartes; however, rather than referring to the intuitive power of the mind, Pascal referred to the heart as the center of intuitive powers. Pascal believed that science is the study of God's world and that the study of mathematics or science for its own sake or for mere enjoyment is a sin. "Such an attitude toward the study of mathematics," he said, "springs from the motivation of exalting self rather than God."

Galileo Galilei (1564-1642) also believed that nature is mathematically designed by God. He believed that the study of nature was on a par with the study of Scriptures as a means of learning the revelation of God. "Nor does God less admirably reveal himself to us in nature's actions that in the scriptures' sacred dictions."

Galileo differed with Descartes in the method of obtaining basic principles. While Descartes believed that the mind supplied the basic principles of nature, Galileo contended that these first principles or basic principles of nature had to be determined experimentally. In criticizing his contemporaries he said, "Nature did not first make men's brains, and then arrange the world so that it would be acceptable to human intellects."

Influenced by the philosophy of the age, Sir Isaac Newton (1642-1727) carried on the conviction that God designed the universe mathematically. He replaced physical hypotheses with mathematical premises which were inferred from observations and experiments. Newton anathematized physics to the extent that mathematics provided the fundamental concepts of physics. He is best known for his work in mechanics and as the co-inventor of calculus; however, religion was the chief motivation for his work in mathematics. Newton, always deeply religious, devoted himself entirely to the study of theology later in his life.

In contrast to Descartes, Newton believed that God intervened now and then to keep the universe functioning according to His plan. Newton had motivation for believing as he did. He had noted that planets did not move in perfectly elliptical orbits. Since he could not account for these irregularities mathematically, and since such perturbations from an elliptical path, he thought, represented deviation from God's original plan, he concluded that God intervened to maintain the stability of His creation.

Leibniz who, you remember, was a contemporary of Newton and co-founder, with Newton, of calculus, felt that this view deprecated God's power. He believed, along with Descartes, that the mechanics of the universe were determined at the time of creation and remain in force invariably.

In the nineteenth century there was a shift in the religious beliefs and philosophies of many intellectuals including mathematicians. This did not happen overnight, nor were the changing attitudes confined to the nineteenth century. Generally speaking, however, around the beginning of the nineteenth century God's role as the designer of the universe was questioned. Though it was not universally the case, mathematicians began to attribute their success in the development of mathematics to their own intellectual powers, and they demanded more rigorous proofs, rather than appealing to God for support of their assertions. Of course, the habit of appealing to God with regard to conjectures which man was unable to establish in any other way was a misguided notion to begin with.

The degeneration of belief in God as the mathematical designer of the universe soon led to the questioning, by a number of mathematicians and philosophers, of laws basic to mathematics. The principles of logic as set down by Aristotle which had been accepted almost without question in the intellectual world for almost two thousand years were being questioned. This was not the first time Aristotelian laws of logic were questioned. Descartes raised the question concerning the validity of the principles of logic. He concluded that they must be true for God would not mislead us. Descartes' conclusion would no longer be considered valid in an intellectual climate in which the very existence of God was being questioned.

In this critical atmosphere, mathematicians became less interested in mathematics as a means to understand nature. The notion of God's mathematical design of the universe was all but dead. The prime concern of mathematicians at the approach of the twentieth century was related to foundational problems in mathematics. This concern for establishing a firm logical foundation for all of mathematics gave rise to several schools of thought. These include the Logicists, Formalists, Intuitionists, and Set Theorists.

Intuitionists believe that mathematics is the product of the human intellect. Historically, Imanuel Kant (1724-1804) was the representative of intuitionism. His chief concern was with a priori knowledge which is independent of experience. He held that our "knowledge of the number rests upon the awareness of the mind of its own capacity to repeat the act of counting time after time." Concerning geometry, he believed that "a priori knowledge rests upon awareness of space as a form of intuition and the mind's awareness of its own capacity to construct spatial figures in imagination." Kant's conception of arithmetic forced him to conclude that numbers exist if and

only if they can be constructed by "counting." Kant, therefore, could not admit Cantor's transfinite numbers. Recall that Cantor assigned the symbol $\chi_{\bullet}$ to the totality of the infinite set consisting of the natural numbers. Then the transfinite number $\chi_{\bullet}$ is assigned to any other infinite set which can be put in a one-to-one correspondence with the set of natural numbers. The number of elements in the infinite set of real numbers turns out to be the same as that of the set of subsets of the natural numbers, since they can be put in a one-to-one correspondence. The symbol that represents the transfinite number of elements in the set of subsets of the natural and thus also represents the cardinality of real numbers is $2^{\chi_{\bullet}}$, which is reminiscent of the symbol used to represent the number of subsets of a set of finite elements. Recall that the number of subsets of a finite set containing "n" elements is 2^n; for example, a set containing three elements would have 2^3 or eight subsets. Kant's doctrine would allow for the potential rather than the actual infinite. The distinction between the potential and the actual infinite were discussed in the last section.

More recently, L. E. Brouwer (1882-1966) became the central figure, extending intuitionism as a philosophical theory to a view that permeated mathematical practices so that there were major disagreements with other mathematicians with respect to the validity of mathematical arguments.

Specifically, Cantor's proof that real numbers cannot be put into a one-to-one correspondence with natural numbers involved the construction of a counter example. Cantor's reasoning went something as follows: If we can establish a one-to-one correspondence between the rational numbers and the real numbers between zero and one, then a one-to-one correspondence has been established between the rationals and all reals, since the real numbers between zero and one would account for every possible infinite sequence containing the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Cantor then gave what high school mathematics students would call an indirect proof. This involves stating an assumption and then providing an argument in which a contradiction involving the assumption is sought. Supposing then that there is such a one-to-one correspondence, we can arrange the real numbers between zero and one in order: $r_1, r_2, r_3, ... r_n ...$ If this arrangement accounts for all the real numbers between zero and one we have an obvious correspondence between the rationals and reals. He then went on to construct a number between zero and one which clearly is not contained in the given sequence. Let this number be called r_0. If the first digit of r_1 is 5, then let the first digit of r_0 be 6; otherwise let the first digit of r_0 be 5. Then, if the second digit of r_2 is 5, assign 6 as the second digit of r_0; otherwise assign 5 as the second digit. Continuing the argument, if the third digit of r_3 is 5, let the third digit of r_0 be 6; if not let the third digit be 5. It is obvious that an endless repetition of such an argument will produce a real number that is not included in the sequence: $r_1, r_2, r_3, ... r_n ...$ Now this number is constructed, using a sequence of "5's" or "6's," and using a rule for assigning these numbers in the sequence. However, this construction is not legitimate according to the intuitionists, since the rule does not show how to construct r_0 through the intuitive activity of counting and calculating. Therefore he rejects Cantor's argument that there are more real numbers than natural numbers. The intuitionist, therefore, considers Cantor's continuum problem invalid, and rejects the whole theory of transfinite numbers.

Cantor's theory of transfinite numbers is not the only victim of the intuitionist's scruples. The intuitionist's philosophy and practices touch many areas of classical mathematics.

Fermat's "last theorem," which we described in the previous section, has neither been proved nor disproved. Now most, but not all, mathematicians who are not intuitionists would claim that the assertion is either true or not true; but this claim assumes the principle of logic called the "law of the excluded middle." The intuitionist, on the other hand, believes that the mind that created numbers must in principle be capable of understanding them completely; thus there can be no unknowable truth or falsity about numbers. Hence, Fermat's "last theorem" is neither true nor false. The intuitionist rejects the "law of the excluded middle." He allows for a middle possibility, that is, that there may well be meaningful statements possessing neither truth nor falsity.

A basic theorem in mathematical analysis asserts that every bounded set of real numbers has a least upper bound. Intuitionists do not accept this theorem because the definition of the least upper bound requires that we mention the set to which it belongs, and is therefore impredicative, which is not allowed by the intuitionist.

Many find intuitionism attractive, for it views mathematical objects as creations of the mind and accords considerable importance to the activity of the mathematician. However, the rejection of certain principles of logic and axioms of classical mathematics seems too high a price to pay to embrace the intuitionistic philosophy.

The intuitionist's philosophy of mathematics could possibly be separated from his mathematical procedures, and it often is in practice; however, if practice is divorced from philosophy, the principles of such practice would be arbitrary and without justification.

In a sense, however, most intuitionists would not be reluctant to make use of theorems, such as the least upper bound theorem, in application, and to do so comfortably, even though, from their view, there may be some inadequacies in the definitions of certain terms involved.

Formalism as opposed to intuitionism maintains that human reason is incapable of producing exact images of geometrical objects or of numbers. Therefore mathematical entities do not have existence in our conception of nature. Formalists and intuitionists agree that mathematics is a product of human thought (they are not Platonists), and they agree that the exact validity of mathematical laws as laws of nature is out of the question. They disagree concerning where the exactness of mathematics does exist. The intuitionist would answer that "mathematical exactness exists in the human intellect," while the formalist would say, "on paper." The formalist claims "mathematical exactness consists merely in the method of developing a series of relations, and is independent of the significance one might want to give the relations or entities which they relate..."

David Hilbert (1862-1943) is most prominently associated with the formalist philosophy. He disagreed with adherents of the logicists school who sought to reduce mathematics to logic. He did, however, agree with Whitehead and Russet concerning the inclusion of infinite sets. Hilbert took exception, to put it mildly, to the intuitionists' view of the infinite, as well as to their rejection of considerable portions of analysis that depended on "existence proofs." He protested that "...to deny existence theorems, such as the existence of the least upper bound of a bounded set, derived by using the principle of the excluded middle, is tantamount to relinquishing the science of mathematics altogether." The intuitionist Weyl said of Hilbert's position that, from the intuitionists' point of view, it is a "bitter but inevitable fact" that only a part, "perhaps only a wretched part," of classical mathematics is tenable. Hilbert could not bear this mutilation. This should give the reader a feeling for the degree of dissension between proponents of the various philosophies of mathematics. Hilbert determined that, in order to avoid the ambiguities of language and to achieve precision and objectivity, all statements of logic and mathematics must be expressed in symbolic form.

The formalist disassociates terms, definitions, axioms, etc. from any mental image. Most of us, because of experience, would have a definite mental image of an axiom such as "two points determine a line." The formalist would not deny us the mental image; however, he would say that such a mental image has no logical import.

Most mathematicians would desire that definitions be "meaningful," in the sense that axioms may be given some interpretation in which they are true. The formalist would claim that the interpretation we give to axioms is irrelevant. The concern is with the logical deductions that result from them. Such results deduced from axioms are theorems which the strict formalist would say need not be interpreted as true or false, since they are based upon axioms which are neither true nor false, for their content originates from undefined terms. We shouldn't make the mistake here of assuming that the formalist rejects the law of the excluded middle. He simply claims that questions of truth or falsity are not appropriate with respect to theorems. All that the formalist can say about his mathematics is that the theorems follow logically from axioms. Then they are free from error. However, as we have noted, certain rules of logic have been questioned by both the logicists and intuitionists.

Formalists' mathematics, in summary, is "a collection of formal systems, each building its own logic along with its mathematics, each having its own concepts, its own rules for deducing theorems, and its own theorems." The development of each of these deductive systems is the task of mathematics. The strict formalist has little concern for interpretation; he believes that mathematics exists and develops for its own sake.

It would appear that the formalist sees mathematics as a game! Such is the charge made by adversaries of the formalist philosophy; and in fact many formalists take exactly this position. Such a position does not take into consideration that it was for the purpose of proving consistency and completeness that formalists reduced mathematics to a series of "meaningless" statements written in symbolic form.

A mathematical system is said to be inconsistent if two contradictory theorems can be deduced from the axioms of the system. A system is said to be complete if every assertion involving concepts of the system can be proved, on the basis of its axioms, to be true or false.

Hilbert, and other mathematicians of the early twentieth century, were confident that it would be possible, eventually, to develop every branch of mathematics in the form of an axiomatic system that could be shown to be both consistent and complete. The ideal was to develop a unified system that could be proved to be both consistent and complete. Their hopes were shattered by Godel in 1931. He proved that for any system that contained even the arithmetic of whole numbers, consistency is incompatible with completeness. That is, if a system is consistent it must necessarily be incomplete. Now this result was disconcerting to the formalists and logicists who had tried to reduce mathematics to logical symbols and principles in order to prove consistency and completeness. The intuitionists, however, contended that this only established what they had believed all along. But the intuitionists had their own problems. As we noted before, there are considerable portions of classical mathematics that have yet to be established using the intuitionist's constructive methods. Mathematicians have yet, despite their philosophical tendencies, to secure a satisfactory foundation for mathematics.

We have sketched the views of the intuitionists and the formalists with characteristics and difficulties peculiar to each. There are other established schools of thought such as the Logicists and Set Theorists. We have alluded to the philosophy of the Logicists. We hope that our discussion of the philosophies of the formalists and intuitionists will serve to acquaint the reader with certain of the modern philosophical problems in mathematics.

It is evident that the concerns of the various modern schools of thought relate to the epistemology of mathematics. Also, there can be no doubt that the philosophies of mathematics influence the way we practice mathematics. It could be said that most practicing mathematicians have little concern for foundational problems; however, every mathematician has a concern for the validity of his work, and realizes that there are certain limits to claims of validity.

Part 3
A CHRISTIAN VIEW OF MATHEMATICS

Often when a Christian perspective of mathematics instruction is sought, reference is made to the orderliness of mathematical systems as somehow expressing the orderliness of God's creation. Though this statement relating, in some way, the orderliness of mathematical systems and the orderliness of creation is vague, the motivation for the statement is clear. Mathematics has been employed for centuries to describe and predict physical phenomena with considerable success.

To even the most casual observer it is obvious that there is a certain predictability in nature. The earth rotates once in approximately 24 hours. It revolves around the sun in approximately 365 days, giving rise to seasons. We in the northern hemisphere realize that it will be colder in December than it is in June, because this has always been the case in our experience. We could continue with many other examples of regularity which suggest order. Because of the degree of success we have experienced in using mathematics to describe and predict physical phenomena, we have tended to ascribe to it a quality of orderliness that corresponds to that of God's creation.

It is our belief that God created the universe according to His own plan, and that the creation took six literal days. We are told in the Scriptures that the creation was complete and perfect in every way, so that there was perfect order and harmony. But sin entered the world and, as a result, God cursed man and the entire physical creation. Just how this curse affected the physical creation we do not know. As far as we can determine, the universe continues to function in a very orderly way. We must recognize that we are limited in our ability to perceive and understand nature. It may very well be that the functioning of the universe is proceeding from harmony and order to disharmony and chaos. We do know that the entire universe will be destroyed by fire at the end of time. We are also told in the Scriptures that existence on this earth will be barely tolerable toward the end. This is not to say that God is losing control of His creation. On the contrary, every event will occur exactly according to His plan. In this sense there is perfect order.

We should insert here that we do not believe Descartes' doctrine that God determined, when He created the universe, every event that should occur and then passively watches the unfolding of His plan. We believe that God is active in carrying out His counsel. This is evident in the Scriptures, where we find many examples of His active involvement. God spoke to Adam and Eve, He also spoke to Cain, and He "walked" with Enoch, with whom He had a special relationship. He spoke to Noah, giving explicit directions concerning the construction of the ark. He "caused" it to rain for forty days and forty nights — an event that could not have been predicted using mathematics or any scientific method. He met with Moses on Mount Sinai, where He gave him commandments. How could we adequately account, using scientific methods, for the manna provided by God for Israel during their trek through the wilderness? And we could elaborate indefinitely, showing that God is active in carrying out His plan. We should not see His activity as "midcourse" corrections made necessary because of flaws in His creation. We should understand this as a sustaining and nurturing activity. The very existence of the entire universe is dependent upon His sustaining power.

According to Descartes and his contemporaries, God designed the universe mathematically. Furthermore, it was held by some, including Descartes, that this mathematical plan was “embedded” in the mind. Others adapted the Platonic philosophy to their belief that God created the universe mathematically. Mathematics then was an intangible part of creation having existence that is independent of the physical world.

We believe that the concept of a mathematically designed universe lacks Scriptural support, as does the philosophy of the Platonists. The idea of mathematics being embedded in the mind is pure speculation. (This idea would receive little support from students who struggle to comprehend basic concepts of mathematics.)

As we have already stated, it is our view that God created the universe according to His own plan. It is presumptuous on our part to impose upon this plan a mathematical design. To do so would almost force us to adopt the philosophy of the Platonists — a view that is inconsistent with the creation account in which we are told specifically what was created on each day. There is no suggestion of the inclusion of intangibles such as various bodies of knowledge.

Another view could be proposed, namely, that mathematics is inherent in the creation of the universe. While we recognize that there are certain intangibles inherent in creation, such as time, distance, etc., we reject the inclusion of any body of knowledge in this category. Time and distance exist as entities apart from the human mind — mathematics does not.

We take the position that ***mathematics should be viewed as an invention of the mind in response to God's creation.*** Therefore, mathematics should never be isolated from the physical world.

This position requires some explanation, for it is by no means universally accepted. We hold to the firm belief in the sovereignty of God — a belief that is rooted in the faith that all of the Scriptures are divinely inspired and infallible. All of the Scriptures testify both implicitly and explicitly of His absolute sovereignty. There is direct reference to this in Matthew 10:29, 30. God is sovereign in the intangible world of human thought and desires. We think, for example, of the instance in which God told Moses to tell Pharaoh to release the Israelites, while at the same time the Lord said that He would “harden” Pharaoh's heart so that he would not let them leave. The Bible teaches that God predestinates so that the events of history are the unfolding of His counsel. However, the Scriptures also teach that we were created in His image; and therefore, we are rational beings, capable of “creative” thought. He speaks to us throughout the Scriptures in a sense that implies that we are capable of thinking, making decisions, and acting upon those decisions. But we do so with constraints. We do not make decisions, for example, concerning our physical makeup or mental capacities. Nor do we make decisions concerning our spiritual orientation. It is altogether reasonable that God, as our creator, would make such determinations. The Scriptures are very specific about this (see Rom. 9:20-23).

There are a few things that we should observe concerning the creations of the human mind.

Our creative acts are not to be understood in the same sense that we perceive creative acts of God. He created the heavens and the physical world out of nothing. He did not have a model upon which He based His creation. Nor did He have “building blocks” to use in some ingenious way in His creation. Our creations, as ingenious as they may be, always use the “building blocks” of past experience and thought. That is, we create in response to God’s work in our lives. David Laverell and Carl Sinke, professors of mathematics at Calvin College, in their paper, “Observations on Mathematics and the Christian Faith,” refer to mathematics as “one facet of man’s response to God’s creation and does not involve the creation of an essentially different world.”

This, we claim, is a distinctively Reformed view. The relationship between man and God is never one in which man acts and God responds, but rather one in which we respond to God’s activity in our lives. Our relationship to Christ, for example, is not that we “invite Christ into our hearts,” resulting in His response to accept the invitation and to save us from our sins. The very opposite is true. God elects to save us and we respond by fearing God and living lives of thankfulness dedicated to His service, though we do not do so perfectly. There are often times inconsistencies between principle and practice in the life of the Christian.

Mathematicians throughout the ages have attempted to be correct; but, as we have pointed out, there have been many errors and blunders both in the development of mathematics and in its external applications. Just as we have but a small and imperfect beginning in our response to God’s redemptive work in our lives, so also our mathematical response to God’s creation is imperfect and only approximates the truth. Though, from our limited perspective, the mathematics we have invented seems very precise and orderly, we cannot claim for it an orderliness on a par with creation.

Our proposal that mathematics is our response to God’s creation is admittedly vague. The word “creation” here is used to refer to the physical universe as it was created by the hand of God, and it also has reference to His activity in that universe throughout history. There is also the spiritual aspect of creation. Though we usually do not think of a mathematics response to this spiritual creation, there are, nevertheless, mathematical concepts with “spiritual overtones,” such as the idea of infinity which we will consider later. Furthermore, it would seem that our spiritual disposition would inform our mathematics, so that our mathematical response to God’s creation would be distinctive. Does this then mean that our mathematical practices and content are different from those of non-Christians?

Before we attempt to answer this question we will consider distinctions between our proposed Christian philosophy and some philosophies that we have discussed.

Our view of mathematics as a response to God’s creation forces us to reject the intuitionist’s constructive philosophy which divorces mathematics from the physical world. Furthermore, the view of the intuitionist that there is neither truth nor falsity associated with a mathematical statement until such time that a constructive proof is formulated implies that truth is time-related. This view of truth is foreign to Christian thought. Sinke and Laverell make a rather strong, but, in our view, an accurate statement concerning this attitude toward truth. “The time dependent quality of truth seems blatantly atheistic. Surely, we say, God knows whether $10^{10^{10}} + 1$ is prime or composite and does not have to wait anxiously for us to perform the calculation.”

Our view of mathematics does not allow us to invest in the formalist’s philosophy. Though we may use formalistic methods in certain applications in mathematics proper, we cannot agree with the philosophy of formalists who view mathematics as nothing more than manipulation of meaningless symbols. The strict formalist, as we have seen, has little concern for interpretation; he believes mathematics exists and develops for its own sake. The mathematics of the Christian must have meaning. We believe in purpose in life, and that we should employ mathematics and all knowledge in pursuit of that purpose, which has to do with enriching our understanding of God through His creation.

Does our philosophy, as we have briefly outlined it, have an effect on our practices? We would have to conclude that it does, for practice is always influenced by philosophy. However, we have to keep in mind that our perspective has to do with why we do mathematics rather than how we do it. Therefore, there is very little that is distinctive about our mathematical methodology.

There are certainly mathematical activities that would be inappropriate for the Christian. There are mathematicians, for example, who have expended enormous effort to prove that man has the potential to create God. This is obviously an atheistic application of mathematics. The Christian mathematician would clearly abstain from such applications and from any mathematical practices and methodology peculiar to this kind of mathematical activity. It could very well be, though, that mathematical practices that are not in themselves evil are used in godless applications such as this.

There are those who believe that applied mathematics is the only proper domain for the Christian mathematician. This is a naive view, born out of the notion that research in the body of mathematics itself is carried out without any external incentives or considerations. Though this may be the attitude of certain purists, it is a mistaken notion, as we have pointed out. It should not be surprising that even the most abstract work of purists finds external applications. It is a fact of history that those works that do not find external application "die on the vine." Mathematics as a body of knowledge, viewed as a response to God's creation, should be continually developed and refined. It is certainly within the bounds of proper stewardship of the Christian mathematician to use his talents in such efforts.

Mathematics is probably the best tool to help us understand the idea of the infinite — a concept which comes to mind when we think of the attributes of God — and a concept which has been at the heart of much of modern mathematics. Mathematicians have, through dint of hard work over a number of years, been able to define mathematical entities, such as the derivative, which involves the concept of infinity. There are many other mathematical concepts which involve the idea of infinity. For example, there are infinitely many real numbers between any two real numbers. Also, the concept of the infinite is used in business applications such as in the calculation of interest compounded continuously. A list of applications of the concept of infinity would itself be endless!

We believe, though this is speculation and not a necessary result of our philosophy, that we are created in a finite setting, i.e., the universe is finite, both with respect to number and size. The universe was created in a finite amount of time, and it will be destroyed in a moment. According to our understanding of the infinite, it seems that, if the physical universe were infinite, the creative process would continue without end and the destructive process would continue without end. But we are quick to admit that we have but a dim comprehension of the infinite. God, Who has no beginning or ending, could very well have created an infinite universe in a moment and consequently could destroy the same in a moment. Nevertheless, in a very real sense we live in a finite setting. Our experience is made up of a finite set of events, each of which has a beginning and an end. However, we believe that each one of us has within him the potential to ponder the infinite, though we do so only with a certain sense of discomfort. Even in our thoughts we cannot escape the finite. Cantor thought to grasp the infinite with his set of all sets, but it eluded him. There is this dimension to our existence of which we can conceive but which we cannot comprehend.

As we have seen, it is difficult to identify mathematical content with the Christian faith. In some cases this is possible, in others it is not possible, or even advisable. For example, part of mathematics includes symbols that are used to help us keep track of our thoughts and to communicate concepts. If our mathematical language did not conform to that used in standard practice it would have very little value. The Christian perspective, though it influences practices and content, has more to do with direction and purpose and therefore has an impact on education in mathematics.

It is not our purpose here to discuss particulars of mathematics instruction. However, a few observations are given for the reader's consideration.

The teacher's role is most often viewed as one in which he "tells" students how to perform various mathematical functions. Such instruction boils down to demonstrating, often in a mechanical way, various techniques which will enable the student to do most of the next day's assignment. This is a very efficient method of teaching if our goal is simply to prepare the student to do a selected list of problems for the following day. In the short run, this method gets an A for efficiency; it might even get an A+ if the list includes some review problems that remind the student of mechanics learned in previous assignments. Another attractive feature of this approach is that it requires less effort on the part of the teacher and student alike. The problem here is that our students have limited memories, just as we do. Long lists of isolated tricks cannot be retained, no matter how often they are reviewed. Some who have extraordinary memories, or who are particularly studious, may proceed happily through most of elementary and high school mathematics before frustration sets in.

We recognize the need to learn the mechanics of mathematics. We would even grant that mathematics cannot be understood if mechanics are not learned. We realize, too, that learning requires a certain amount of rote drill. However, we maintain that if learning mechanics through rote drill is the goal, then we have not taught mathematics, nor has the student learned mathematics.

We propose that, in order to make any meaningful mathematical response to God's creation, we must understand mathematical concepts. This provides us with a goal in our instruction. In order to accomplish this goal, we must first of all teach the necessary mechanics. In addition to this, meaning must be given to mathematical processes so that students learn concepts and relationships between those concepts. This is not a simple task; it requires more effort on the part of the teacher and the student.

In order to accomplish this, we suggest that mathematics be taught from a historical perspective. By this we do not necessarily mean that the teacher reveal names and dates and other details of the historical setting of a given concept, though such information should be included whenever possible. Teaching from a historical perspective requires fist of all that we show how a concept or process is developed and, whenever possible, explain relationships between various concepts and processes.

Instruction of this type would necessarily be interactive. We suggest the following general approach.

Initially, the teacher would be responsible for developing and explaining the concept at the students' level of understanding. The student would then be required to respond by constructing his understanding of the concept. This would be followed by a period of reflection in which the student would attempt to determine if his construct coincides with that of the teacher. Finally, the teacher and student would together make a judgment concerning the adequacy of the student's construction of the concept. This may result in some alterations in the student's construct, and further reflection by the student. In summary, this approach requires the teacher initially to explain a concept. The student then constructs, and in concert with the teacher reconstructs, his understanding of the concept until both the student and the teacher are satisfied with the student's level of understanding.

Our students should have an awareness of the fact that the mathematics that they see in their textbooks is the result of individual and combined effort of mathematicians throughout history, and that they were many mistakes made along the way. It is important for students to realize that even the greatest and most powerful intellects in mathematics have made errors. This is necessary, we believe, since many students fear mathematics because they have the mistaken notion that mathematics is infallible, and the people who do mathematics never err.

We have rejected the Platonist's philosophy of mathematics, yet we often teach as Platonists, and therefore students see mathematics from a Platonist's view — as a body of knowledge, perfect and polished, residing in textbooks. It appears to them to have existence entirely apart from any human input. Eventually they learn that this is not the case, but first impressions die hard.

The idea of the conceptual approach to teaching mathematics is not new. Most mathematics teachers agree with this ideal. Many, however, feel that this ideal is unattainable. There are several reasons for this. In the rest place, teaching concepts requires more effort on the part of the teachers. Secondly, students resist learning concepts. And finally, most teachers assume that the conceptual approach requires more time than is available.

We agree that more effort is required. But for teachers who have a thorough knowledge of mathematics, the extra effort should be both enjoyable and rewarding. Students' resistance would cease if teachers at all levels were consistent in using the conceptual approach. Teaching conceptually need not require more time. The ideal that we have presented can be realized. However, it would require a higher level of intensity and devotion on the part of teachers and students.

If this approach is not used, our students, or to make this more personal, our covenant young people, end up holding the bag. And in that bag are innumerable, isolated tricks and devices used to complete assignments. The contents of this bag may represent a kind of mathematics, but it is not the mathematics that satisfies our Christian philosophy nor our Christian commitment to learning.

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